Series Review 2 _ solutions

Note Title 2/7/2012

(1) $f(x) = \cos(2x)$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

$$= 1 - 2x^2 + \frac{16}{24}x^4 - \frac{64}{720}x^6 + \dots$$

$$= 1 - 2x^2 + \boxed{\frac{2}{3}x^4} \dots$$

$\boxed{C}$

(2) $\sum_{1}^{\infty} \frac{(x-2)^n}{n(-3)^n}$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+1\ (-3)^{n+1}} \cdot \frac{n(-3)^n}{(x-2)^n} \right| = \left| \frac{x-2}{-3} \right| < 1$$

$$|x-2| < 3$$

$\boxed{(-1, 5]}$ $\boxed{C}$

Test end pts.

$\frac{(-1-2)^n}{n(-3)^n} = \cancel{\frac{-3^n}{-3^n}} \cdot \frac{1}{n}$ Diverges Harmonic

$\frac{(5-2)^n}{n(-3)^n} = \frac{(-1)^n}{n}$ Converges Alt. Harmonic

(3) $x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \frac{x^{2n+1}}{n!}$

$$\frac{x \cdot x^{2n}}{n!} = \frac{x \cdot (x^2)^n}{n!} = x \cdot e^{x^2}$$ $\boxed{D}$

(4) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2} + \frac{(2x)^3}{3!}$$

$$\frac{8x^3}{6} = \frac{4}{3}x^3$$

$\boxed{D}$

(5) $f(x) = \sum_{n=1}^{\infty} (\cos x)^{3n}$

$$f\left(\frac{2\pi}{3}\right) = \sum_{n=1}^{\infty} \left(\cos \frac{2\pi}{3}\right)^{3n} = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{3n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{-1}{8}\right)^n$$

$$= -\frac{1}{8} + \frac{1}{64} - \frac{1}{8^3} \cdots$$

$$\frac{-\frac{1}{8}}{1 - -\frac{1}{8}} = \frac{-\frac{1}{8}}{\frac{9}{8}} = \boxed{-\frac{1}{9}}$$

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⑥ $\frac{9}{8} - \frac{3}{4} + \frac{1}{2} - \frac{1}{3} + \cdots$ ratio $= -\frac{2}{3}$

$$\frac{\frac{9}{8}}{1 - -\frac{2}{3}} = \frac{\frac{9}{8}}{\frac{5}{3}} = \frac{9}{8} \cdot \frac{3}{5} = \boxed{\frac{27}{40}}$$

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⑦

n	$f^n(x)$	$f^n(0)$	$f^n(0)/n!$
0	$(1+x)^{1/2}$	1	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$	$-\frac{1}{8}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$	$\frac{1}{16}$

$$\boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3}$$

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⑧ ⓐ $1 + \frac{2!}{3}(x-2) + \frac{3!}{2!\,3^2}(x-2)^2 + \frac{4!}{3!\,3^3}(x-2)^3 + \cdots \frac{(n+1)!}{n!\,3^n}(x-2)^n$

$$\boxed{1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots + \frac{n+1}{3^n}(x-2)^n}$$

ⓑ $\lim_{n\to\infty}\left|\frac{(n+2)(x-2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)(x-2)^n}\right| = \left|\frac{x-2}{3}\right| < 1$

$|x-2| < 3$

$\boxed{\text{Radius of conv. } = 3}$

ⓒ $\int g' = \int 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \cdots \frac{n+1}{3^n}(x-2)^n$

$$g(x) = C + x + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \frac{1}{3^3}(x-2)^4 + \ldots \quad \frac{1}{3^n}(x-2)^{n+1}$$

$$g(2) = 3 = C + 2 + 0 + 0 + 0 \ldots$$
$$1 = c$$

$$g(x) = 1 + x + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \ldots + \frac{1}{3^n}(x-2)^{n+1}$$

ⓓ $$\lim_{n\to\infty}\left|\frac{(x-2)^{n+2}}{3^{n+1}} \cdot \frac{3^n}{(x-2)^{n+1}}\right| = \left|\frac{x-2}{3}\right| < 1$$

$$|x-2| < 3$$

No, the series for g will not converge for $x=-2$ because -2 lies beyond of the reach of the radius of convergence.

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⑨

n	$f^n(x)$	$f^n(0)$	$f^n(0)/n!$
0	$\cos(3x+\frac{3\pi}{4})$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
1	$-3\sin(\ )$	$-\frac{3\sqrt{2}}{2}$	$-\frac{3\sqrt{2}}{2}$
2	$-9\cos(\ )$	$\frac{9\sqrt{2}}{2}$	$\frac{9\sqrt{2}}{2\cdot 2!}$
3	$27\sin(\ )$	$\frac{27\sqrt{2}}{2}$	$\frac{27\sqrt{2}}{2\cdot 3!}$

$$\boxed{P(x) = -\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}x + \frac{9\sqrt{2}}{2\cdot 2!}x^2 + \frac{27\sqrt{2}}{2\cdot 3!}x^3}$$

ⓑ $$\left|f(\tfrac{1}{6}) - P(\tfrac{1}{6})\right| = R_3(\tfrac{1}{6}) \le \left|\frac{f^4(z)}{4!}(\tfrac{1}{6}-0)^4\right|$$

$f^4(z)$ is max of $81\cos(3x+\frac{3\pi}{4})$ on $(0, \frac{1}{6})$ $= 81$

$$\le \left|\frac{81}{4!\, 6^4}\right| = \frac{27}{4\cdot 2\cdot 6^4}$$
$$= \frac{\cancel{3\cdot 3\cdot 3}}{4\cdot 2\cdot \cancel{3}\cdot 2\cdot \cancel{3}\cdot 2\cdot \cancel{3}\cdot 2\cdot 6}$$
$$= \frac{1}{8\cdot 4\cdot 12}$$
$$= \frac{1}{384} < \frac{1}{300}$$

and there was much rejoicing!

ⓒ $$G(x) = \int_0^x f(t)\,dt$$

$$= \int_0^x -\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}t + \frac{9\sqrt{2}}{2\cdot 2!}t^2 \quad dt$$

$$= \boxed{-\frac{\sqrt{2}}{2}x - \frac{3\sqrt{2}}{4}x^2 + \frac{3\sqrt{2}}{4}x^3}$$