Differentiation by the Chain Rule - Classwork

Suppose you were asked to take the derivatives of the following. Do so.

a) $f(x)=(x^2+3x)^2$ b) $f(x)=(x^2+3x)^3$ c. $f(x)=(x^2+3x)^6$ d) $f(x)=\sqrt{x^2+3x}$

a) causes no problem. b) is also not a problem but multiplying it out so you can take the derivative is a bit of a pain. You are capable of doing c) but clearly do not wish to. But d) can't be done with the knowledge you have.

We now introduce a method of taking derivatives of more complicated expressions. It is called the **chain rule**. If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentable function of x, then $y=f(g(x))$ is a differentable function of x and $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ or equivalently, $\frac{d}{dx}[f(g(x))]=f'(g(x))\cdot g'(x)$

Example 1) If $f(x)=(2x+5)^2$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule b) with chain rule

Example 2) If $f(x)=(2x+5)^3$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule b) with chain rule

Example 3) If $f(x)=(2x+5)^{10}$, find $f'(x)$

Example 4) If $f(x)=\sqrt{2x+5}$, find $f'(x)$

Example 5) Find y' if $y=\frac{1}{4x-3}$

Example 6) Find y' if $y=(3x^2-2x+1)^3$

Find the derivatives of the following:

7) $y=\left(7-4x^2\right)^{2/3}$

8) $y=-5\sqrt{x^2-4x+1}$

9) $y=\dfrac{-2}{\sqrt[4]{6x-1}}$

More difficult problems: We now have 3 basic rules. Power rule, product rule, and quotient rule. Note that the chain rule is not a basic rule of differentiation. The chain rule is always in effect. Even when you find the derivative of $y = 7x$, your answer is 7 times the derivative of x which is $7(1) = 7$.

Find the derivatives of the following:

10) $y=x^2\left(2x-3\right)^4$

11) $y=x\sqrt{4-x^2}$

12) $y=\left(\dfrac{2x-1}{2x+1}\right)^5$

13) $y=\dfrac{x}{\sqrt{x^2-1}}$

14) $y=\sqrt{\dfrac{x}{4x-1}}$

15) $y=\left(x^2-4\right)\sqrt{x+2}$

Given that $f(2)=-3, f'(2)=6, g(2)=3, g'(2)=-2, f'(3)=4$, find the derivatives of the following at $x = 2$.

16) $f(x)\cdot g(x)$

17) $\dfrac{f(x)}{g(x)}$

18) $\left[f(x)\right]^3$

19) $f(g(x))$

Differentiation by the Chain Rule - Homework

Find the derivatives of the following:

1. $y=(3x-8)^4$

2. $y=(3x^2+2)^5$

3. $y=4(x^2+x-1)^{10}$

4. $y=-5(4-9x)^{3/2}$

5. $y=\dfrac{1}{3x-2}$

6. $y=\dfrac{-1}{(x^2-5x-6)^2}$

7. $y=\left(\dfrac{2}{2-x}\right)^2$

8. $y=\dfrac{4x}{(x+1)^2}$

9. $y=\dfrac{-3}{(x^3-x^2+3)^3}$

10. $y=x^3(5x-1)^4$

11. $y=\sqrt{1-t}$

12. $y=\sqrt[3]{3x^3-4x+2}$

13. $y=\dfrac{2}{\sqrt{2x+3}}$

14. $y=\dfrac{-1}{\sqrt{x}+1}$

15. $y=\sqrt{\dfrac{3x}{2x-3}}$

16. $y = \sqrt{x}(1-2x)^2$

17. $y = \sqrt[3]{\dfrac{2t}{t^2-4}}$

18. $y = (x^2+2x-6)^2(1-x^3)^2$

For each of the following, find the equation of the tangent line at the indicated point. Verify by calculator.

19. $y = \sqrt{x^2+2x+8}$ at $(2,4)$

20. $y = \sqrt[5]{3x^3+4x}$ at $(2,2)$

21. $y = \sqrt{\dfrac{3x-1}{2x+1}}$ at $(-1,2)$

Given the following information, find the value of the derivative of the functions at $x = 3$. Be careful, not all the information is needed to calculate these. Answers are next to the problem.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	π	4
1	2	-6	5	0

22. $f(x)+g(x)$ (Ans: -8)

23. $f(x)g(x)$ (Ans: -29)

24. $\dfrac{f(x)}{g(x)}$ (Ans: $\dfrac{-19}{64}$)

25. $\dfrac{g(x)}{f(x)}$ (Ans: 19)

26. $(f(x))^2$ (Ans: -6)

27. $\dfrac{1}{g(x)}$ (Ans: $\dfrac{5}{64}$)

28. $\sqrt{f(x)}$ (Ans: $\dfrac{-3}{2}$)

29. $\sqrt{f(x)+g(x)}$ (Ans: $\dfrac{-4}{3}$)

30 $f^3(x)g(x)$ (Ans: -77)

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	π	4
1	2	-6	5	0

31. $\frac{1}{\sqrt[3]{g(x)}}$ (Ans: $\frac{5}{48}$)

32. $\frac{f(x)}{f(x)+g(x)}$ (Ans: $\frac{-19}{81}$)

33. $f(g(x))$ (Ans: -5π)

34. $g(f(x))$ (Ans: 0)

35. $f(f(x))$ (Ans: -15)

36. $g(g(x))$ (Ans: -20)

37. The table below gives some values of the derivative of some function *f*. Complete the table by finding (if possible) the derivatives of each of the following transformations of *f*.

a) $g(x)=f(x)-2$

b) $h(x)=2f(x)$

c) $r(x)=f(-3x)$

d) $s(x)=f(2x+1)$

x	**-2**	**-1**	**0**	**1**	**2**	**3**
$f'(x)$	4	$2/3$	$-1/3$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

1. Complete the table.

$y=f(g(x))$	$y=f(u)$	$u=g(x)$
a.) $y=\frac{1}{\sqrt{x+1}}$		
b.) $y=3\tan\left(\pi x^2\right)$		
c.) $y=\csc^3 x$		

2. Find the derivative function.

a.) $y=2\sqrt[4]{4-x^2}$ b.) $y=-\frac{5}{(t+3)^3}$ c.) $g(t)=\sqrt{\frac{1}{t^2-2}}$ d.) $f(x)=x(3x-9)^3$

e.) $y=\frac{x}{\sqrt{x^4+4}}$ f.) $g(x)=\left(\frac{3x^2-2}{2x+3}\right)^3$ g.) $y=\sin\sqrt[3]{x}+\sqrt[3]{\sin x}$

3. Evaluate the derivative of the function at the indicated point.

a.) $y=\sqrt[5]{3x^3+4x}$ $(2,2)$ b.) $y=37-\sec^3(2x)$ $(0,36)$ c.) $y=\frac{1}{x}+\sqrt{\cos x}$ $\left(\frac{\pi}{2},\frac{2}{\pi}\right)$

4. Find the equation of the tangent line to the graph of f at the indicated point.

a.) $f(x)=\frac{1}{3}x\sqrt{x^2+5}$ $(2,2)$ b.) $f(x)=\tan^2 x$ $\left(\frac{\pi}{4},1\right)$

5. Given the functions defined below, complete the table.

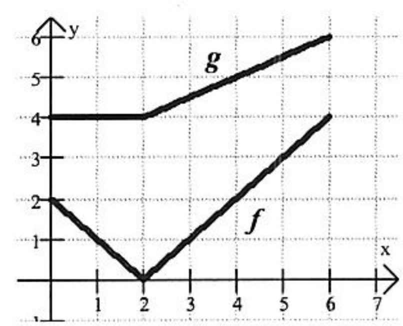
$g(x)=f(x)-2$
$h(x)=2f(x)$
$r(x)=f(-3x)$
$s(x)=f(x+2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

6. Let $r(x)=f(g(x))$ and $s(x)=g(f(x))$ where f and g are shown at the right. Find:

a.) $r'(1)$ b.) $s'(4)$

c.) $h'(1)$ if $h(x)=f^3(x)+g^2(x)$ d.) $h'(4)$ if $h(x)=\sqrt{g(x)}$



7. Show that the derivative of an odd function is even. That is if $f(-x)=-f(x)$, then $f'(-x)=f'(x)$.

8. Determine whether each statement is <u>true or false</u>. If it is false, explain why or give a counter example.

a.) If $y=(1-x)^{\frac{1}{2}}$, then $y'=\frac{1}{2}(1-x)^{-\frac{1}{2}}$.

b.) If $f(x)=\sin^2(2x)$, then $f'(x)=2(\sin 2x)(\cos 2x)$

c.) If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then $\frac{dy}{dx}=\frac{dy}{du}\square\frac{du}{dv}\square\frac{dv}{dx}$

AP Calculus AB / RBHS Fun Worksheet 3.5 Solutions ☺

1a.) $y=\frac{1}{\sqrt{u}}$ $u=x+1$ 1b.) $y=3\tan u$ $u=\pi x^2$ 1c.) $y=u^3$ $u=\csc x$

2a.) $y'=\frac{-x}{\sqrt[4]{\left(4-x^2\right)^3}}$ 2b.) $y'=\frac{15}{(t+3)^4}$ 2c.) $g'(t)=\frac{-t}{\sqrt{\left(t^2-2\right)^3}}$

2d.) $f'(x)=(3x-9)^2(12x-9)$ 2e.) $\frac{dy}{dx}=\frac{4-x^4}{\left(x^4+4\right)^{3/2}}$ 2f.) $g'(x)=\frac{3\left(3x^2-2\right)^2\left(6x^2+18x+4\right)}{(2x+3)^4}$

2g.) $\frac{dy}{dx}=\cos\sqrt[3]{x}\left(\frac{1}{3\sqrt[3]{x^2}}\right)+\frac{\cos x}{3\sqrt[3]{\sin^2 x}}$

3a.) $\frac{1}{2}$ 3b.) 0 3c.) $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}}=\text{DNE}$

4a.) $y-2=\frac{13}{9}(x-2)$ 4b.) $y-1=4\left(x-\frac{\pi}{4}\right)$

5)

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$	*dne*	12	1	*dne*	*dne*	*dne*
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4	*dne*	*dne*

6a.) 0 6b.) DNE 6c.) -3 6d.) $\frac{1}{4\sqrt{5}}$

7) $\frac{d}{dx}[f(-x)]=\frac{d}{dx}[-f(x)]\Rightarrow -f'(-x)=-f'(x)\Rightarrow f'(-x)=f'(x)$

8a.) False $y'=-\frac{1}{2}(1-x)^{-1/2}$ 8b.) False $f'(x)=4\sin 2x\cos 2x$ 8c.) True

Worksheet 3.1-3.4

1. Evaluate the following.

a.) $\lim\limits_{\Delta x\to 0}\dfrac{5(x+\Delta x)^4-\dfrac{3}{(x+\Delta x)}-5x^4+\dfrac{3}{x}}{\Delta x}$

b.) $\lim\limits_{h\to 0}\dfrac{\tan(\frac{5\pi}{4}+h)-1}{h}$

c.) $\lim\limits_{h\to 0}\dfrac{6\sqrt[3]{\frac{1}{8}+h}-6\sqrt[3]{\frac{1}{8}}}{h}$

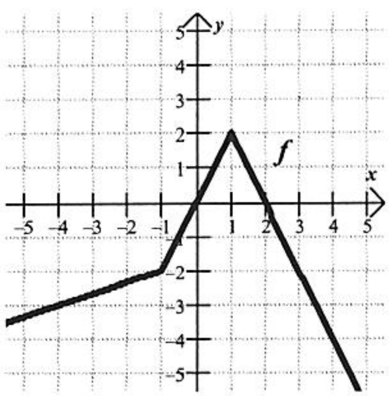
2. Find the value of k such that the curve $y=k\sqrt{x}-32$ and line $y=2x$ are tangent at a common point.

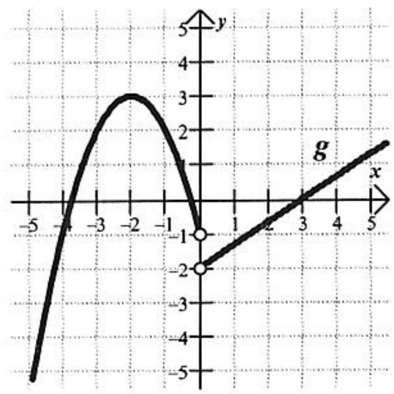
3. Find all values of x on $[0,2\pi)$ where $f(x)=\tan x+\sec x$ has:

a) Vertical asymptotes

b) Horizontal tangent lines

4. Use the graphs of f and g pictured below to determine the following.





a.) If $P(x)=6f(x)-4g(x)$, find $P'(-3)$.

b.) If $H(x)=\dfrac{1}{f(x)}$, find $H'(3)$.

c.) If $R(x)=\dfrac{f(x)}{g(x)}$, find $R'(-2)$

d.) If $y=\sqrt[3]{x^2}\,g(x)$, find $\left.\dfrac{dy}{dx}\right|_{x=1}$

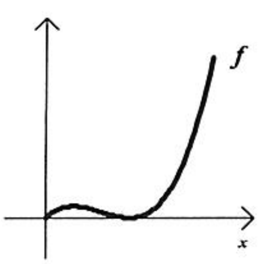
e.) At $x = -1$ is f differentiable? Justify your answer.

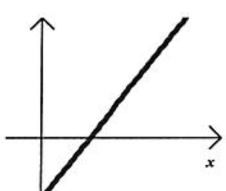
f.) Given the graph of g, sketch a reasonable graph of $g'(x)$ on the interval $(-5,5)$.

5. Find a quadratic function with a y-intercept of 10 and whose tangent line at the point $(5\ ,0)$ has a slope of 8.

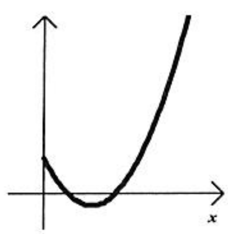
6. Given $f(x)=\begin{cases}x^2+kx-3, & x<1\\ 3x+b, & x\geq 1\end{cases}$, find k and b such that $f(x)$ is differentiable at $x = 1$.

7. Given below is the graph of a function f. Beside this graph are three others that are, in random order, the graphs of f', f'', and a function F whose derivative is f ($F'=f$). Label each graph with the correct function.

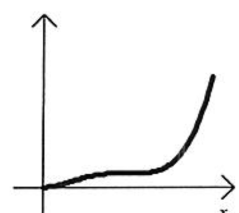




Function __________



Function __________

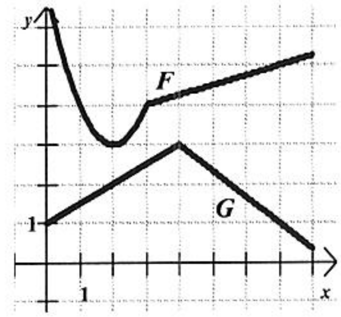


Function __________

8. Let f and g be functions that are differentiable for all real numbers x with $g(x) = \frac{f(x)}{x}$. If $y = 2x - 3$ is an equation of the line tangent to the graph of f at $x = 1$, what is the equation of the line tangent to the graph of g at $x = 1$?

9. Find equations of both tangent lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

10. Use the graphs of F and G pictured at the right to determine the following.



a.) If $P(x) = G(x)F(x)$, find $P'(2)$.

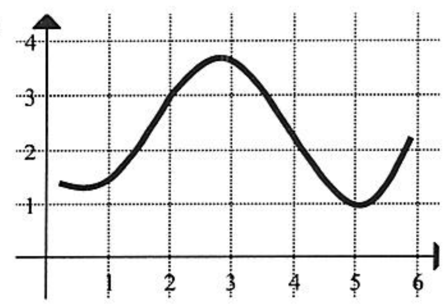
b.) If $Q(x) = \frac{G(x)}{F(x)}$, find $Q'(7)$.

c.) If $y = x^2 F(x)$, find $\left.\frac{dy}{dx}\right|_{x=2}$

11. Evaluate $\lim_{h \to 0} \frac{(1+h)^7 - 1}{h}$.

12. If $f(x) = x^4 + ax^2 + 8x - 5$ has a horizontal tangent at $x = 1$, then $a = ?$

13. Given the graph of $f(x)$, determine whether each of the following is true or false.

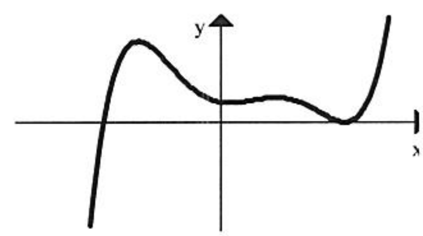


a.) $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 0$

b.) $\frac{f(5) - f(2)}{5 - 2} = -\frac{2}{3}$

c.) $f'(2) > f'(5)$

14. The graph of a fifth-degree polynomial $f(x)$ is shown to the right. The graph of $f'(x)$, the derivative of $f(x)$, will cross the x-axis in exactly how many places?



15. If $g(x) = x^3 f(x)$, find $g'(2)$ if $f(2) = 6$ and $f'(2) = 3$.

16. If the tangent line to $y = \frac{kx + 8}{k + x}$ at $x = -2$ is $y = x + 4$, find k.

17. Find the values for x where $y = x^2 + x + \frac{1}{2}$ and $x + 2y = 3$ intersect at right angles to each other.

18. If $3x - y + 2 = 0$ is tangent to $f(x) = x^3 + k$ in Quadrant I, find k.

Worksheet 3.1-3.4 Solutions

1. a.) $f'(x) = 20x^3 + \dfrac{3}{x^2}$ b.) 2 c.) 8

2. $k=16$ -

3. a.) x= $\dfrac{\pi}{2}$, $\left(x = \dfrac{3\pi}{2} \text{ is a hole} \right)$ b.) None

4.a.) $P'(-3) = -6$. b.) $-\dfrac{1}{2}$

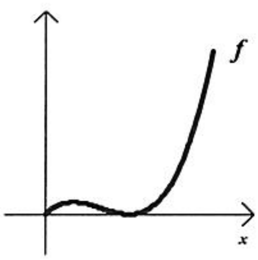
c.) $R'(-2) = \dfrac{1}{9}$ d.) $\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{-2}{9}$

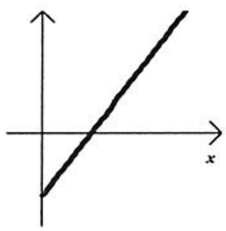
e.) No , $f_-'(-1) \neq f_+'(-1)$

5. $y = 2x^2 - 12x + 10$

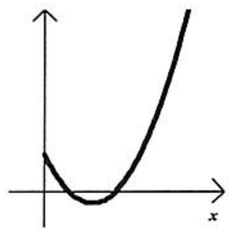
6. k= 1 and b =-4

7.

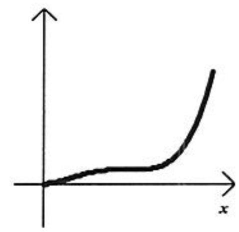




Function ___ f'' ____



Function ___ f' ____



Function ___ $F' = f$ ____

8. $y = 3x - 4$

9. 11(x-5) = y -30 and -1(x+1) = y

10. a.) $P'(2) = \dfrac{3}{2}$. b.) $Q'(7) = \dfrac{-43}{300}$. c.) $\left.\dfrac{dy}{dx}\right|_{x=3} = 12$

11. $f'(1) = 7$

12. a = -6

13. a.) False b.) True c.) True

14. 4

15. 96

16. k= 3

17. x =1/2

18. k=4