

Differentiation by the Chain Rule - Classwork

Suppose you were asked to take the derivatives of the following. Do so.

a) $f(x) = (x^2 + 3x)^2$ b) $f(x) = (x^2 + 3x)^3$ c) $f(x) = (x^2 + 3x)^6$ d) $f(x) = \sqrt{x^2 + 3x}$

$f(x) = x^4 + 6x^3 + 9x^2$ $f(x) = x^6 + 9x^5 + 27x^4 + 27x^3$ Forget it cannot do it
 $f'(x) = 4x^3 + 18x^2 + 18x$ $f'(x) = 6x^5 + 45x^4 + 108x^3 + 81x^2$

a) causes no problem. b) is also not a problem but multiplying it out so you can take the derivative is a bit of a pain. You are capable of doing c) but clearly do not wish to. But d) can't be done with the knowledge you have.

We now introduce a method of taking derivatives of more complicated expressions. It is called the **chain rule**. If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then $y = f(g(x))$ is a differentiable function of x and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ or equivalently, $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Example 1) If $f(x) = (2x+5)^2$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule

$$f(x) = 4x^2 + 20x + 25$$

$$f'(x) = 8x + 20$$

b) with chain rule

$$f'(x) = 2(2x+5)(2)$$

$$f'(x) = 8x + 20$$

Example 2) If $f(x) = (2x+5)^3$, find $f'(x)$ without and with the chain rule. Show they are equivalent.

a) without chain rule

$$f(x) = 8x^3 + 60x^2 + 150x + 125$$

$$f'(x) = 24x^2 + 120x + 150$$

b) with chain rule

$$f'(x) = 3(2x+5)^2(2)$$

$$f'(x) = 6(2x+5)^2$$

$$f'(x) = 24x^2 + 120x + 150$$

Example 3) If $f(x) = (2x+5)^{10}$, find $f'(x)$

$$f'(x) = 10(2x+5)^9(2)$$

$$f'(x) = 20(2x+5)^9$$

Example 4) If $f(x) = \sqrt{2x+5}$, find $f'(x)$

$$f(x) = (2x+5)^{1/2}$$

$$f'(x) = \frac{1}{2}(2x+5)^{-1/2}(2)$$

$$f'(x) = \frac{1}{(2x+5)^{1/2}}$$

Example 5) Find y' if $y = \frac{1}{4x-3}$

$$\frac{dy}{dx} = -(4x-3)^{-2}(4) = \frac{-4}{(4x-3)^2}$$

Example 6) Find y' if $y = (3x^2 - 2x + 1)^3$

$$y' = 3(3x^2 - 2x + 1)^2(6x - 2)$$

Find the derivatives of the following:

$$7) y = (7 - 4x^2)^{2/3}$$

$$y' = \frac{2}{3}(7 - 4x^2)^{-1/3}(-8x)$$

$$y' = \frac{-16x}{3(7 - 4x^2)^{1/3}}$$

$$8) y = -5\sqrt{x^2 - 4x + 1}$$

$$y' = \frac{-5(2x - 4)}{2(x^2 - 4x + 1)^{1/2}}$$

$$y' = \frac{-5(x - 2)}{(x^2 - 4x + 1)^{1/2}}$$

$$9) y = \frac{-2}{\sqrt[4]{6x - 1}}$$

$$y' = \frac{3}{(6x - 1)^{5/4}}$$

More difficult problems: We now have 3 basic rules. Power rule, product rule, and quotient rule. Note that the chain rule is not a basic rule of differentiation. The chain rule is always in effect. Even when you find the derivative of $y = 7x$, your answer is 7 times the derivative of x which is $7(1) = 7$.

Find the derivatives of the following:

$$10) y = x^2(2x - 3)^4$$

$$11) y = x\sqrt{4 - x^2}$$

$$12) y = \left(\frac{2x - 1}{2x + 1}\right)^5$$

$$y' = 6x(2x - 3)^3(2x - 1)$$

$$y' = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}}$$

$$y' = \frac{20(2x - 1)^4}{(2x + 1)^6}$$

$$13) y = \frac{x}{\sqrt{x^2 - 1}}$$

$$14) y = \sqrt{\frac{x}{4x - 1}}$$

$$15) y = (x^2 - 4)\sqrt{x + 2}$$

$$y' = \frac{-1}{(x^2 + 1)^{3/2}}$$

$$y' = \frac{-1}{2(4x - 1)^2} \left(\frac{4x - 1}{x}\right)^{1/2}$$

$$y' = \frac{(x^2 - 4)}{2(x + 2)^{1/2}} + 2x(x + 2)^{1/2}$$

Given that $f(2) = -3, f'(2) = 6, g(2) = 3, g'(2) = -2, f'(3) = 4$, find the derivatives of the following at $x = 2$.

$$16) f(x) \cdot g(x)$$

$$17) \frac{f(x)}{g(x)}$$

$$18) [f(x)]^3$$

$$19) f(g(x))$$

$$\begin{aligned} f(x)g'(x) + g(x)f'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \\ f(2)g'(2) + g(2)f'(2) &= \frac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} \\ -3(-2) + 3(6) = 24 &= \frac{3(6) - (-3)(-2)}{3^2} = \frac{4}{3} \end{aligned}$$

$$3[f(x)]^2 f'(x)$$

$$3[f(2)]^2 f'(2)$$

$$3(-3)^2(6) = 162$$

$$f'[g(x)] \cdot g'(x)$$

$$f'[g(2)] \cdot g'(2)$$

$$f'(3)(-2)$$

$$4(-2) = -8$$

Differentiation by the Chain Rule - Homework

Find the derivatives of the following:

1. $y = (3x-8)^4$

2. $y = (3x^2 + 2)^5$

3. $y = 4(x^2 + x - 1)^{10}$

$$y' = 12(3x - 8)^3$$

$$y' = 30x(3x^2 + 2)^4$$

$$y' = 40(x^2 + x - 1)^9(2x + 1)$$

4. $y = -5(4-9x)^{3/2}$

5. $y = \frac{1}{3x-2}$

6. $y = \frac{-1}{(x^2 - 5x - 6)^2}$

$$y' = \frac{135(4-9x)^{1/2}}{2}$$

$$y' = \frac{-3}{(3x-2)^2}$$

$$y' = \frac{2(2x-5)}{(x^2 - 5x - 6)^3}$$

7. $y = \left(\frac{2}{2-x}\right)^2$

8. $y = \frac{4x}{(x+1)^2}$

9. $y = \frac{-3}{(x^3 - x^2 + 3)^3}$

$$y' = \frac{8}{(2-x)^3}$$

$$y' = \frac{-4x+4}{(x+1)^3}$$

$$y' = \frac{27x^2 - 18x}{(x^3 - x^2 + 3)^4}$$

10. $y = x^3(5x-1)^4$

11. $y = \sqrt{1-t}$

12. $y = \sqrt[3]{3x^3 - 4x + 2}$

$$y' = x^2(5x-1)^3(35x-3)$$

$$y' = \frac{-1}{2\sqrt{1-t}}$$

$$y' = \frac{9x^2 - 4}{3(3x^3 - 4x + 2)^{2/3}}$$

13. $y = \frac{2}{\sqrt{2x+3}}$

14. $y = \frac{-1}{\sqrt{x+1}}$

15. $y = \sqrt{\frac{3x}{2x-3}}$

$$y' = \frac{-2}{(2x+3)^{3/2}}$$

$$y' = \frac{1}{2\sqrt{x}(\sqrt{x+1})^2}$$

$$y' = \frac{-9}{2(2x-3)^2} \left(\frac{2x-3}{3x}\right)^{1/2}$$

16. $y = \sqrt{x(1-2x)}^2$

17. $y = \sqrt[3]{\frac{2t}{t^2-4}}$

18. $y = (x^2 + 2x - 6)^2(1 - x^3)^2$

$$y' = \frac{20x^2 - 12x + 1}{2\sqrt{x}}$$

$$y' = \frac{-2t^2 - 8t}{3(t^2 - 4)^2} \left(\frac{t^2 - 4}{2t} \right)^{2/3}$$

$$y' = -6x^2(x^2 + 2x - 6)^2(1 - x^3) + (1 - x^3)^2(4x + 4)(x^2 + 2x - 6)$$

For each of the following, find the equation of the tangent line at the indicated point. Verify by calculator.

19. $y = \sqrt{x^2 + 2x + 8}$ at $(2, 4)$

20. $y = \sqrt[5]{3x^3 + 4x}$ at $(2, 2)$

21. $y = \sqrt{\frac{3x-1}{2x+1}}$ at $(-1, 2)$

$$y' = \frac{x+1}{\sqrt{x^2 + 2x + 8}}$$

$$y' = \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}}$$

$$y' = \frac{1}{2} \sqrt{\frac{2x+1}{3x-1}} \left(\frac{5}{(2x+1)^2} \right)$$

$$y'(2) = \frac{3}{4}$$

$$y'(2) = \frac{1}{2}$$

$$y'(-1) = \frac{5}{4}$$

$$y - 4 = \frac{3}{4}(x - 2)$$

$$y - 2 = \frac{1}{2}(x - 2)$$

$$y - 2 = \frac{5}{4}(x + 1)$$

Given the following information, find the value of the derivative of the functions at $x = 3$. Be careful, not all the information is needed to calculate these. Answers are next to the problem.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	π	4
1	2	-6	5	0

22. $f(x) + g(x)$ (Ans: -8)

23. $f(x)g(x)$ (Ans: -29)

24. $\frac{f(x)}{g(x)}$ (Ans: $-\frac{19}{64}$)

25. $\frac{g(x)}{f(x)}$ (Ans: 19)

26. $(f(x))^2$ (Ans: -6)

27. $\frac{1}{g(x)}$ (Ans: $\frac{5}{64}$)

28. $\sqrt{f(x)}$ (Ans: $-\frac{3}{2}$)

29. $\sqrt{f(x) + g(x)}$ (Ans: $-\frac{4}{3}$)

30. $f^3(x)g(x)$ (Ans: -77)

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	π	4
1	2	-6	5	0

31. $\frac{1}{\sqrt[3]{g(x)}}$ (Ans: $\frac{5}{48}$)

32. $\frac{f(x)}{f(x)+g(x)}$ (Ans: $\frac{-19}{81}$)

33. $f(g(x))$ (Ans: -5π)

34. $g(f(x))$ (Ans: 0)

35. $f(f(x))$ (Ans: -15)

36. $g(g(x))$ (Ans: -20)

37. The table below gives some values of the derivative of some function f . Complete the table by finding (if possible) the derivatives of each of the following transformations of f .

a) $g(x) = f(x) - 2$

b) $h(x) = 2f(x)$

c) $r(x) = f(-3x)$

d) $s(x) = f(2x+1)$

x	-2	-1	0	1	2	3
$f(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$???	12	1	???	???	???
$s'(x)$???	$\frac{4}{3}$	-2	-8	???	???