Exponential Growth - Homework

1. Major purchases like cars depreciate in value. That is, as time goes on, their value goes down. So the change in a car's price is directly proportional to its current value.

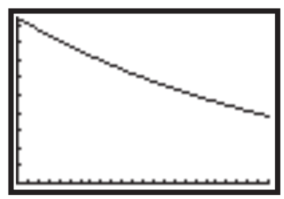
   a) Write a differential equation that expresses this relationship, and then solve the DEQ for the value of the car.

   $$\frac{dP}{dt} = kP \Rightarrow P = Ce^{kt}$$

   b) Suppose you own a car whose trade-in value is presently \$4200. Three months ago, its trade-in value was \$4700. Find the particular equation that expressed the trade-in value since the car was worth \$4700.

   $$P = 4700e^{kt} \Rightarrow 4200 = 4700e^{3k} \Rightarrow k = \ln\left(\frac{42}{47}\right)\Big/3 = -.037$$

   c) Sketch the graph of this equation over 2 years.



   d) What will be the trade-in value of the car one year after the car was worth \$4700?

   $$P(12) = 2997.12$$

   e) You plan to get rid of the car when its trade-in value drops to \$1200. When will this be?

   $$1200 = 4700e^{kt}$$

   $$t = \ln\left(\frac{12}{47}\right)\Big/k \approx 36 \text{ months}$$

   f) At the time your car was worth \$4700, it was 31 months old. What was its trade-in value when it was new?

   $$P(-31) = 15{,}026.80$$

   g) The purchase price of the car when it was new was \$16,000. How do you explain the difference between this number and f)?

   Car loses value when it is driven.

2. Carbon 14 is an isotope of carbon that is formed when radiation from the sun strikes ordinary carbon dioxide in the atmosphere. Thus plants such as trees, which get their carbon dioxide from the atmosphere, contains small parts of carbon 14. The rate of carbon 14 decay is directly proportional to the amount present.

   a) Write a differential equation that expresses this relationship where P is the percentage of carbon 14 that remains in a tree that grew t years ago.

   $$\frac{dP}{dt} = kP \Rightarrow P = Ce^{kt}$$

   b) Solve the differential equation for P in terms of t. Use the fact that the half-life of carbon 14 is 5750 years to solve the particular equation for the constant P_0. (If $P = 100$ when $t = 0$, then $P = 50$ when $t = 5750$).

   $$50 = 100e^{5750k} \Rightarrow k = \ln .5/5750$$

c) The oldest living trees in the world are the bristlecone pines in the White Mountains of California. 4000 growth rings have been counted in the trunk of one of these trees, meaning that the innermost ring grew 4000 years ago. What percentage of the original carbon 14 would you expect to find in this innermost ring?

$$P(4000) = 61.74\%$$

d) A piece of wood claimed to have come from Noah's Ark is found to have 48.37% of the carbon 14 remaining. It has been suggested that the Great Flood occurred in 4004 BC. Is the wood old enough to have come from Noah's Ark? Show why or why not.

$$48.37 = 100e^{kt} \Rightarrow t = 6024 \text{ years ago} \Rightarrow \text{it could be old enough}$$

3. The rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant. At 15°C, the pressure is 101.3 at sea level and 87.1 at height h = 1000m.

   a) Write a differential equation that expresses this relationship where P is the pressure and h is the altitude. Then solve the particular DEQ given the information above.

$$\frac{dP}{dh} = kP \Rightarrow P = 101.3e^{kh} \Rightarrow k = \ln\left(\frac{87.1}{101.3}\right) \Big/ 1000$$

   b) What is the pressure at the top of Mt. McKinley at an altitude of 6187 meters?

$$P(6187) = 39.792$$

   c) At what altitude is the pressure equal to 50?

$$50 = 101.3e^{kh} \Rightarrow h = \ln\left(\frac{50}{101.3}\right) \Big/ k = 4675 \text{ m}$$

4. You accidently inhale some poisonous fumes. Twenty hours later, you still feel woozy so you go to a doctor. From blood samples, he measures a poison concentration of .00372 mg/ml and tells you to come back in 8 hours. On the second visit, he measures a concentration of .00219 mg/ml.

   Let t be the number of hours since your first doctor visit and C be the concentration of poison in your blood. The rate of change of C is directly proportional to the current value of C.

   a) Write a DEQ that relates these two variables.

   b) Solve the differential equation subject to initial conditions specified. Express C as a function of t.

$$\frac{dC}{dt} = kC \Rightarrow P = .00372e^{kt} \Rightarrow k = \ln\left(\frac{.00219}{.00372}\right) \Big/ 8 = -.066$$

   c) The doctor says that you might have serious body damage if the poison concentration has ever been as high as 0.015 mg/ml. Based on your equation, was the concentration ever that high? Justify your answer.

$$C(-20) = .0139 \text{ - Never as high}$$

5. The rate in which whooping cough spreads is proportional to the amount of whooping cough cases there presently is. In the course of any year, the number of whooping cough cases is reduced by 20%.

a) Write a DEQ that states the situation above and then solve the DEQ.

$$80 = 100e^{k} \Rightarrow k = \ln .8 = -.223$$

b) If there are 10,000 cases today, how many years will it take for the number of cases to reduce to 100?

$$100 = 10000e^{kt} \Rightarrow t = \ln .01/k = 20.638 \text{ yrs}$$

c) If the cases can be reduced by 25% instead of 20%, how long will it take to reduce to 100 cases?

$$k = \ln .75 \Rightarrow 100 = 10000e^{kt} \Rightarrow t = \ln .01/k = 16.008 \text{ yrs}$$

d) Using part c), how long will it take to eradicate the disease (reduce # of cases to less than 1)?

$$1 = 10000e^{kt} \Rightarrow t = \ln .0001/k \approx 32 \text{ yrs}$$

6. John Napier, who invented natural logarithms, was the first person to answer the question: If money compounds at a rate proportional to the amount you now have, and you invest money at 5% interest,

a) Write and solve the DEQ which states the above.

$$\frac{dM}{dt} = kM \Rightarrow M = M_0e^{kt}$$

$$1.05 = 1.00e^{k} \Rightarrow k = \ln(1.05)$$

b) how long will it take \$100 to grow to \$1,000?

$$1000 = 100e^{kt}$$

$$t = \frac{\ln 10}{k} = 47.19 \text{ yrs}$$

c) how long will it take any amount of money to triple?

$$3M = Me^{kt} \Rightarrow t = \ln 3 / k = 22.517 \text{ yrs}$$

c) if you invest \$1,000, when will you be a millionaire?

$$1000000 = 1000e^{kt}$$

$$t = \frac{\ln 1000}{k} = 141.581 \text{ yrs}$$

7. The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of sugar is proportional to the amount of raw sugar remaining. If 1,000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much sugar will remain after another 12 hours?

$$800 = 1000e^{10k} \Rightarrow k = \ln .8/10 = -.022$$

$$P = 1000e^{kt} \Rightarrow P(22) = 612.066 \text{ kg}$$

Exponential Growth Continuation - Classwork

Remember that you can always translate $\frac{dy}{dt} = ky$ **into** $y = Ce^{kt}$ if and only if the original statement is:
The rate of change of some quantity y is directly proportional to y. Suppose it isn't? For example:

a) the rate of change of *y* is proportional to 4*y*.

$$\frac{dy}{dt} = 4ky \Rightarrow \int \frac{dy}{y} = \int 4kdt$$
$$\ln|y| = 4kt + C \Rightarrow y = Ce^{4kt}$$

b) the rate of change of *y* is proportional to 4 - *y*.

$$\frac{dy}{dt} = k(4-y) \Rightarrow \int \frac{dy}{4-y} = \int kdt$$
$$-\ln|4-y| = kt + C \Rightarrow 4 - y = Ce^{kt}$$
$$y = 4 - Ce^{kt}$$

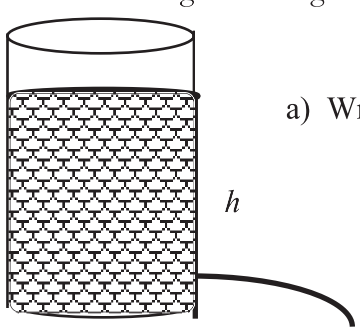
c) the rate of change of *y* is inversely proportional to *y*

$$\frac{dy}{dt} = \frac{k}{y} \Rightarrow \int y\,dy = \int k\,dt$$
$$\frac{y^2}{2} = kt + C \Rightarrow y^2 = 2kt + C$$

d) the rate of change of *y* is proportional to $\sqrt{y}$

$$\frac{dy}{dt} = k\sqrt{y} \Rightarrow \int y^{-1/2}\,dy = \int k\,dt$$
$$2y^{1/2} = kt + C \Rightarrow y^{1/2} = \frac{kt}{2} + C$$
$$y = \left(\frac{kt}{2} + C\right)^2$$

Example. Suppose you fill a tall tin can with water and then punch out a hole near the bottom. The water leaks out quickly at first, then more slowly as the depth of the water decreases. In physics, it can be proved that the rate at which the water's height *h* changes (i.e. leaks out) is directly proportional to the square root of its height.



a) Write a differential equation which states this relationship.

$$\frac{dh}{dt} = k\sqrt{h}$$

b) Suppose that at time $t = 0$ min, the height is 12 cm and $dh/dt = -3$ cm/sec. Find the value of *k* which satisfies this relationship.

$$\frac{dh}{dt} = k\sqrt{h} \Rightarrow -3 = k\sqrt{12} \Rightarrow k = \frac{-\sqrt{3}}{2}$$

c) Solve this differential equation to find *h* as a function of *t*. Use the given information to find the particular solution. What kind of function is this?

$$\frac{dh}{dt} = -\frac{\sqrt{3}}{2}\sqrt{h} \Rightarrow \int h^{-1/2}\,dh = \int -\frac{\sqrt{3}}{2}dt \Rightarrow 2h^{1/2} = -\frac{\sqrt{3}}{2}t + C \Rightarrow h^{1/2} = -\frac{\sqrt{3}}{4}t + C$$
$$h = \left(-\frac{\sqrt{3}}{4}t + C\right)^2 \Rightarrow C = \sqrt{12} \Rightarrow h = \left(-\frac{\sqrt{3}}{4}t + \sqrt{12}\right)^2 \Rightarrow h = \frac{3}{16}t^2 - 3t + 12$$

d) Solve algebraically for the time when the can becomes empty.

$$-\frac{\sqrt{3}}{4}t + \sqrt{12} = 0 \Rightarrow \sqrt{3}t = 4\sqrt{12} \Rightarrow t = 8 \text{ sec}$$

Exponential Growth Continuation - Homework

1. A curve passes through the point (0, 5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

$$\frac{dy}{dt} = 2y \Rightarrow \int \frac{dy}{y} = \int 2\,dt$$

$$\ln|y| = 2t + C \Rightarrow y = Ce^{2t} \Rightarrow y = 5e^{2t}$$

2. Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and the outside air. Suppose that a roast turkey is taken from the oven when its temperature has reached 185° and is placed on a table where the temperature is 75°. If R is the temperature of the turkey after t minutes, then Newton's Law of Cooling implies that:

$$\frac{dR}{dt} = k(R - 75)$$

a) Solve the differential equation for R. Then use the given information to find the particular solution.

$$\frac{dR}{dt} = k(R - 75) \Rightarrow \frac{dR}{(R - 75)} = k\,dt$$

$$\ln|R - 75| = kt + C \Rightarrow R - 75 = Ce^{kt}$$

$$R = 75 + Ce^{kt} \Rightarrow 185 = 75 + C \Rightarrow C = 110$$

$$R = 75 + 110e^{kt}$$

b) If the temperature of the turkey is 150° after half an hour, what is the temperature after 45 minutes?

$$150 = 75 + 110e^{30k} \Rightarrow k = \ln\frac{75}{110}\Big/30$$

$$R(45) = 136.929^{o}$$

c) When will the turkey have cooled to 100°?

$$100 = 75 + 110e^{kt} \Rightarrow t = \ln\frac{25}{110}\Big/k \Rightarrow k = 116.055 \text{ min}$$

3. The change of rate of coyotes N in a population is directly proportional to 650 - N. (where t is the time in years. When t = 0, the population is 300 and when t = 2, the population has increased to 500.

a) Write and solve a differential equation that describes this situation.

$$\frac{dN}{dt} = k(650 - N) \Rightarrow \frac{dN}{(650 - N)} = k\,dt \Rightarrow -\ln|650 - N| = kt + C \Rightarrow 650 - N = Ce^{-kt}$$

$$N = 650 - Ce^{-kt} \Rightarrow 300 = 650 - C \Rightarrow C = 350 \Rightarrow N = 650 - 350e^{-kt} \Rightarrow 500 = 650 - 350e^{-2k}$$

$$k = \ln\frac{150}{350}\Big/-2 \approx .424$$

b. Find the coyote population in 3 years

$$N(3) \approx 552$$

c. Find $\lim_{t \to \infty} N(t)$

650

4. Let $P(t)$ represent the number of students in a school who buy their lunch after t weeks. Suppose P is increasing at a rate proportionally to $600 - P$ where the constant of proportionality is k.

a) Write the DEQ which states that fact. $\frac{dP}{dt} = k(600 - P)$

b) If 300 students buy their lunch initially and 400 buy their lunches after 10 weeks, solve the DEQ.

$$\frac{dN}{dt} = k(600 - N) \Rightarrow \int \frac{dP}{600 - N} = \int k\,dt \Rightarrow -\ln|600 - P| = kt + C \Rightarrow 600 - P = Ce^{-kt}$$

$$P = 600 - Ce^{-kt} \Rightarrow 300 = 600 - C \Rightarrow C = 300 \Rightarrow P = 600 - 300e^{-kt} \Rightarrow 400 = 600 - 300e^{-10k}$$

$$k = \ln\frac{200}{300} \Big/ -10 \approx .041$$

c) How many students will buy their lunch after 20 weeks?

$$P(20) \approx 467$$

d) If school went on endlessly using this pattern, what is the limit to the number of students buying lunch?

$$\lim_{t \to \infty} P(t) = 600$$

5) You win a well-known national sweepstakes. Your award is $100 a day for the rest of your life! You put the money in a bank where it earns interest at a rate directly proportional to the amount M which is in the account. So, $\frac{dM}{dt} = 100 + kM$ where k is the growth constant.

a) Solve the DEQ in general given the fact that at $t = 0$ days, there is no money in the account.

$$\frac{dM}{dt} = 100 + kM \Rightarrow \frac{1}{k}\int \frac{k\,dM}{100 + kM} = \int dt \Rightarrow \frac{1}{k}\ln|100 + kM| = t + C \Rightarrow 100 + kM = Ce^{kt}$$

$$kM = Ce^{kt} - 100 \Rightarrow M = \frac{Ce^{kt} - 100}{k} \Rightarrow 0 = \frac{C - 100}{k} \Rightarrow C = 100 \Rightarrow M = \frac{100e^{kt} - 100}{k}$$

b) Suppose you invest the money at 5% APR So $k = \frac{.05}{365}$. Solve the DEQ completely.

$$M(t) = \frac{365\left(100e^{0001369t} - 100\right)}{.05}$$

c) How much will you have at the end of a year?

$$M(365) = \$37427.90$$

d) Assuming you live for 75 more years, how much will you take to the grave with you if you never spend it?

$$M(75 \cdot 365) = \$30,310,389.86$$

e) How long will it take you to become a millionaire? How about a billionaire? Calculators allowed.

$$\frac{100e^{kt} - 100}{k} = 1000000 \text{ - Solve Graphically} \Rightarrow t = 6298.67 \text{ days} = 17.26 \text{ years}$$

$$\frac{100e^{kt} - 100}{k} = 1000000000 \text{ - Solve Graphically} \Rightarrow t = 52729.33 \text{ days} = 144.46 \text{ years}$$