arctan or *ln* ???

Some of these problems are integrated as *arctan*, while others are *ln*. In the end, they get pretty tricky!

1. $\int \frac{dx}{4+9x^2}$

2. $\int \frac{x}{4+x^2}dx$

3. $\int \frac{x^4+x^3}{x^2+1}dx$

4. $\int \frac{1}{x(3+\ln x)}dx$

5. $\int \frac{7-\ln x}{x(3+\ln x)}dx$

6. $\int \frac{1}{x^2+4x+5}dx$

7. $\int \frac{1}{2x^2+12x+68}dx$

8. $\int \frac{1}{\sqrt{x}(x+9)}dx$

9. $\int \frac{\cos x}{1+\sin^2 x}dx$

10. $\int \frac{e^{2x}}{1+e^{2x}}dx$

11. $\int \frac{e^{x}}{1+e^{2x}}dx$

12. $\int \frac{e^{4x}}{1+e^{2x}}dx$

13. $\int \frac{x}{x^2+2x+5}dx$

14. $\int \frac{1}{1+e^{2x}}dx$

15. $\int \frac{e^{3x}}{1+e^{2x}}dx$

arctan or ln???

① $\displaystyle\int \frac{dx}{4+9x^2} = \frac{1}{6}\arctan\frac{3x}{2} + C$

$u = 3x \quad du = 3dx$

$a = 2$

② $\displaystyle\int \frac{x}{4+x^2}\,dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln(4+x^2) + C$

$4+x^2 = u$

$du = 2x\,dx$

③ $\displaystyle\int \frac{x^4+x^3}{x^2+1}\,dx = \int (x^2+x-1)\,dx - \int\frac{x}{x^2+1}\,dx + \int\frac{dx}{x^2+1} =$

$u = x^2+1 \quad du = 2x\,dx$

$$\begin{array}{r}
x^2 + x - 1 + \frac{-x+1}{x^2+1} \\
x^2+1 \overline{\big)\, x^4 + x^3 \qquad} \\
-(x^4 \qquad + x^2) \\
\hline
x^3 - x^2 \\
-(x^3 \qquad + x) \\
\hline
-x^2 - x \\
-(-x^2 - 1) \\
\hline
-x + 1
\end{array}$$

$\displaystyle\frac{x^3}{3} + \frac{x^2}{2} - x - \frac{1}{2}\int\frac{du}{u} + \arctan x$

$\displaystyle\frac{x^3}{3} + \frac{x^2}{2} - x - \frac{1}{2}\ln(x^2+1) + \arctan x + C$

④ $\displaystyle\int \frac{1}{x(3+\ln x)}\,dx = \int\frac{du}{u} = \ln|3+\ln x| + C$

$u = 3 + \ln x$

$du = \frac{1}{x}\,dx$

⑤ $\displaystyle\int \frac{7-\ln x}{x(3+\ln x)}\,dx = \int\frac{7}{x(3+\ln x)}\,dx - \int\frac{\ln x}{x(3+\ln x)}\,dx$

$u = 3+\ln x \qquad \ln x = u - 3$

$du = \frac{1}{x}dx$

$\displaystyle 7\int\frac{du}{u} - \int\frac{u-3}{u}\,du = 7\int\frac{du}{u} - \int 1 - \frac{3}{u}\,du =$

$7\ln|u| - u + 3\ln|u| = 10\ln|3+\ln x| - 3 - \ln x + C_1$

$= 10\ln|3+\ln x| - \ln x + C$

(6) $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{x^2+4x+4+1} = \int \frac{dx}{(x+2)^2+1} = \boxed{\arctan(x+2) + C}$

(7) $\int \frac{dx}{2(x^2+6x+34)} = \frac{1}{2}\int \frac{dx}{x^2+6x+9+25} = \frac{1}{2}\int \frac{dx}{(x+3)^2+5^2} = \frac{1}{2}\left[\frac{1}{5}\arctan\frac{x+3}{5}\right]$

$= \boxed{\frac{1}{10}\arctan\frac{x+3}{5} + C}$

(8) $\int \frac{1}{\sqrt{x}(x+9)}dx = 2\int \frac{du}{u^2+9} = 2\left[\frac{1}{3}\arctan\frac{\sqrt{x}}{3}\right] = \boxed{\frac{2}{3}\arctan\frac{\sqrt{x}}{3} + C}$

$u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}dx$

(9) $\int \frac{\cos x}{1+\sin^2 x}dx = \int \frac{du}{1+u^2} = \boxed{\arctan(\sin x) + C}$

$u = \sin x$

$du = \cos x\, dx$

(10) $\int \frac{e^{2x}}{1+e^{2x}} = \frac{1}{2}\int \frac{du}{u} = \boxed{\frac{1}{2}\ln(1+e^{2x}) + C}$

$u = 1+e^{2x}$

$du = 2e^{2x}dx$

(11) $\int \frac{e^x}{1+e^{2x}}dx = \int \frac{du}{1+u^2} = \boxed{\arctan e^x + C}$

$u = e^x$

$du = e^x dx$

(12) $\int \frac{e^{4x}}{1+e^{2x}}dx = \int e^{2x}dx - \int \frac{e^{2x}dx}{e^{2x}+1} = \frac{1}{2}\int e^u du - \frac{1}{2}\int \frac{e^u}{e^u+1}du$

$v = e^u+1$

$dv = e^u du$

$$\begin{array}{r} e^{2x} - \frac{e^{2x}}{e^{2x}+1} \\ e^{2x}+1 \overline{)\, e^{4x} \qquad\qquad} \\ -(e^{4x}+e^{2x}) \\ \hline -e^{2x} \end{array}$$

$2x = u$

$du = 2dx$

$= \frac{1}{2}e^{2x} - \frac{1}{2}\ln|e^u+1|$

$= \boxed{\frac{1}{2}e^{2x} - \frac{1}{2}\ln(1+e^{2x}) + C}$

(13) $\displaystyle\int \frac{x}{x^2+2x+5}\,dx = \int \frac{x+1-1}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx - \int \frac{dx}{x^2+2x+5} =$

$u = x^2+2x+5$

$du = (2x+2)\,dx$

$\displaystyle\frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx - \int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\ln(x^2+2x+5) - \frac{1}{2}\arctan\frac{x+1}{2} + C$

(14) $\displaystyle\int \frac{1}{1+e^{2x}}\,dx = \int \frac{1}{1+e^{2x}}\cdot\frac{e^{-2x}}{e^{-2x}}\,dx = \int \frac{e^{-x}\cdot e^{-x}}{e^{-2x}+1}\,dx = -\int \frac{u\,du}{u^2+1} = -\frac{1}{2}\int \frac{dv}{v}$

$u = e^{-x}$ $\qquad$ $v = u^2+1$

$du = -e^{-x}\,dx$ $\qquad$ $dv = 2u\,du$

$\displaystyle= -\frac{1}{2}\ln\left|e^{-2x}+1\right| = -\frac{1}{2}\ln\left(\frac{1}{e^{2x}}+1\right) + C$

(15) $\displaystyle\int \frac{e^{3x}}{1+e^{2x}}\,dx = \int e^x\,dx - \int \frac{e^x\,dx}{1+e^{2x}} = e^x - \int \frac{du}{1+u^2} =$

$$\begin{array}{r} e^x - \dfrac{e^x}{e^{2x}+1} \\ e^{2x}+1 \,\big)\, e^{3x} \\ -(e^{3x}+e^x) \\ \hline -e^x \end{array}$$

$e^x = u$

$du = e^x\,dx$

$e^x - \arctan e^x + C$