Review Sheet #5 Solutions

① $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} x^{2n+3}}{(2n+2)!} \cdot \frac{(2n)!}{\cancel{(-1)^n} x^{2n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0 \qquad R=\infty \qquad (-\infty,\infty)$$

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② ⓐ $\lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}}{(n+1)^2} \cdot \frac{n^2}{\cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \cdot x \right| = |x| < 1 \qquad [-1,1]$

pdf Key is wrong!

$\sum \frac{(-1)^{n+1}}{(n+1)^2}$ ✓ Alternating Series

$\sum \frac{(1)^{n+1}}{(n+1)^2}$ ✓ limit comparison $\lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = 1$ ✓

ⓑ $f(x) = \frac{x}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \dots \frac{x^{n+1}}{(n+1)^2}$

$f'(x) = 1 + \frac{2x}{4} + \frac{3x^2}{9} + \frac{4x^3}{16} + \quad \frac{(n+1)x^n}{(n+1)^2}$

$= 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots \frac{x^n}{n+1}$

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③ ⓐ $f(x) = \frac{1}{1+x^2}$ Think $\frac{1}{1+x}$ and sub in x^2

$\hookrightarrow 1 - x + x^2 - x^3 + x^4 + \dots (-1)^n x^n$

$1 - x^2 + (x^2)^2 - (x^2)^3 + (x^2)^4 + \dots (-1)^n x^{2n}$

$\boxed{1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n}}$

ⓑ $g(x) = \arctan x$ that's the antiderivative!

$g(x) = \int 1 - x^2 + x^4 - x^6 + \dots (-1)^n x^{2n}\, dx$

$g(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \frac{(-1)^n x^{2n+1}}{2n+1}$

$g(0) = 0 = C +$ a bunch of zeros ; so $C = 0$

$\boxed{g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \frac{(-1)^n x^{2n+1}}{2n+1}}$

ⓒ $g(\tfrac{1}{3}) = \frac{1}{3} - \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} - \frac{(\frac{1}{3})^7}{7} + \dots$ for $R_N \leq .001$ we want first omitted term to be $< \frac{1}{1000}$

$= \frac{1}{3} - \frac{1}{3^3 \cdot 3} + \underbrace{\frac{1}{3^5 5}}_{\frac{1}{1215}} - \frac{1}{3^7 7}$

this is error term. Actually, this is more alternating series remainder than the LaGrange Form ...

$\frac{1}{3} - \frac{1}{81} = \frac{27}{81} - \frac{1}{81} = \boxed{\frac{26}{81}}$

(4) $\sin(x^3) = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \frac{(x^3)^7}{7!} + \dots \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$

$$\boxed{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \dots \frac{(-1)^n x^{6n+3}}{(2n+1)!}}$$

(5) $\frac{x}{1+2x} = x\left(\frac{1}{1+(2x)}\right) = x\left(1 - (2x) + (2x)^2 - (2x)^3 + \dots (-1)^n (2x)^n\right)$

$= x\left(1 - 2x + 4x^2 - 8x^3 + \dots (-1)^n (2x)^n\right)$

$= \boxed{x - 2x^2 + 4x^3 - 8x^4 + \dots (-1)^n 2^n x^{n+1}}$

(6) $\ln(3-x)$ $c=2$

n	$f^n(x)$	$f^n(2)$	$\frac{f^n(2)}{n!}$
0	$\ln(3-x)$	0	0
1	$\frac{-1}{3-x}$	-1	-1
2	$\frac{-1}{(3-x)^2}$	-1	$\frac{-1}{2}$
3	$\frac{-2}{(3-x)^3}$	-2	$\frac{-1}{3}$
4	$\frac{-6}{(3-x)^4}$	-6	$-\frac{1}{4}$

$-1(x-2) - \frac{1}{2}(x-2)^2 - \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \dots \frac{-(x-2)^{n+1}}{n+1}$

(7) $f(x) \approx P_5(x) = 2x - 5x^3 + 4x^5$

$f' = 2 - 15x^2 + 20x^4$

$f'' = -30x + 80x^3$

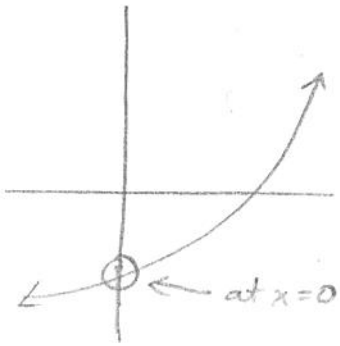
$f''' = -30 + 240x^2$

$f^{4} = 480x$

$f^5 = 480$

$$\boxed{\begin{array}{l} f''(0) = 0 \\ f'''(0) = -30 \\ \\ f^5(0) = 480 \end{array}}$$

(8) f is increasing and concave up; so f' has to be + and f'' has to be positive



at x=0 f is −, f' is +, f'' is +

n	$\frac{f^n(0)}{n!}$	
0	a	← negative
1	b	← positive
2	c	← positive

(9) $1 + \frac{3}{1!} + \frac{9}{2!} + \frac{27}{3!} + \quad \frac{3^n}{n!}$ This is $\frac{x^n}{n!}$ ... ooh, baby! That's e^x so, $\boxed{e^3}$

(10) $1 + \left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^3 + \ldots$

This is geometric ...

Also ... $1 + x + x^2 + x^3 + \ldots x^n = \frac{1}{1-x}$ $\qquad \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \boxed{\frac{5}{4}}$

(11) $\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \frac{(-1)^n x^n}{n!}\right)}{x}$

$= \lim_{x \to 0} \frac{2x + 2\frac{x^3}{3!} + \frac{2x^5}{5!} + \ldots \frac{2x^{2n+1}}{(2n+1)!}}{x}$

$= \lim_{x \to 0} \frac{\cancel{x}\left(2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \ldots \frac{2x^{2n}}{(2n+1)!}\right)}{\cancel{x}}$

$= \boxed{2}$

sweet!

(12) $\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots \frac{(-1)^n x^{2n+1}}{(2n+1)!}}{x}\,dx$

$= \int_0^1 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} + \ldots \frac{(-1)^n x^{2n}}{(2n+1)!}\,dx$

$= x - \frac{x^3}{3!\cdot 3} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!} + \ldots \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \Big|_0^1$

$= \boxed{1 - \frac{1}{3!\cdot 3} + \frac{1}{5\cdot 5!}} - \frac{1}{7\cdot 7!}$

$1 - \frac{1}{18} + \frac{1}{600} \approx .946111$

Since $\int_0^1 \frac{\sin x}{x}\,dx$ is an alternating series, the error is the first omitted term.

the fourth term $\frac{1}{7\cdot 7!} = \frac{1}{35280}$ is the first term $< \frac{1}{1000}$.

(13)

(a)

n	$f^n(3)$	$\frac{f^n(3)}{n!}$
0	-5	-5
1	2	2
2	-7	$-7/2$
3	9	$3/2$

$\boxed{P_3(x) = -5 + 2(x-3) - \frac{7}{2}(x-3)^2 + \frac{3}{2}(x-3)^3}$

$P_3(2.6) = -5 + 2(-.4) - \frac{7}{2}(-.4)^2 + \frac{3}{2}(-.4)^3 \approx \boxed{-6.456}$

(13 b) $|f(2.6) - P_3(2.6)| \le \left|\frac{f^{4}(z)}{4!}(2.6-3)^4\right|$

$\le \frac{5}{4!}(-.4)^4$

$\le .005333$ (max error)

so $f(2.6)$ has be ~~in~~ interval

$P_3(2.6) - R_3(2.6) \le f(2.6) \le P_3(2.6) + R_3$

$-6.46133 \le f(2.6) \le -6.45067$

-6 is not on this interval!

$f(2.6) \ne 6$

(c)

n	$g^n(x)$	$g^n(0)$	$\frac{g^n(0)}{n!}$
0	$f(x^2+3)$	-5	-5
1	$2x f'(x^2+3)$	0	0
2	see (2)	4	2
3	(3)	0	0
4	(4)	-84	$\frac{-84}{24}$

$2x\, f'(x^2+3)$

$(2x)(2x) f''(x^2+3) + f'(x^2+3)\cdot 2$

$4x^2 f''(x^2+3) + 2f'(x^2+3) \leftarrow$ (2)

$(2x)4x^2 f'''(x^2+3) + f''(x^2+3)(8x) + 2f''(x^2+3)\cdot 2x$ (3)

$8x^3 f'''(x^2+3) + 8x\, f''(x^2+3) + 4x\, f''(x^2+3)$

$(2x)8x^3 f''''(x^2+3) + f'''(x^2+3)(24x^2) + 8x\, f'''(x^2+3)(2x) + f''(x^2+3)\cdot 8$
$+ 4x\, f'''(x^2+3)\cdot(2x) + f''(x^2+3)\cdot 4$ (4)

$\boxed{-5 + 2x^2 - \frac{7}{2}x^4}$ yowza!

$-7\cdot 8 + -7\cdot 4$

$-56 \quad -28$

(d)

$g(x) = -5 + 2x^2 - \frac{7}{2}x^4$

$g'(x) = 4x - 14x^3$ so $x=0$ is a critical pt!

$g''(x) = 4 - 42x^2$

$g''(0) = 4$ since $g''(0) > 0$, concave up, meaning g(0) is a relative minimum

(14)

n	$f^n(4)$	$\frac{f^n(4)}{n!}$
0	1	1
1	$\frac{-1}{3(2)}$	$\frac{-1}{3(2)}$
2	$\frac{2!}{3^2(3)}$	$\frac{\cancel{2!}}{3^2(3)\cancel{2!}} = \frac{1}{3^2(3)}$
3	$\frac{-3!}{3^3(4)}$	$\frac{-\cancel{3!}}{3^3(4)\cancel{3!}} = \frac{-1}{3^3(4)}$

$f^n(4) = \frac{(-1)^n\, n!}{3^n(n+1)}$

$\boxed{1 - \frac{1}{3(2)}(x-4) + \frac{1}{3^2(3)}(x-4)^2 - \frac{1}{3^3(4)}(x-4)^3}$

$+ \ldots \frac{(-1)^n}{3^n(n+1)}(x-4)^n$

(b) $\sum_{n=0} \frac{(-1)^n (x-4)^n}{3^n (n+1)}$

$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} (x-4)^{\cancel{n+1}}}{3^{n+1} (n+2)} \cdot \frac{3^{\cancel{n}} (n+1)}{\cancel{(-1)^n} \cancel{(x-4)^n}} \right|$

$\lim_{n\to\infty} \left| \frac{x-4}{3} \cdot \frac{n+1}{n+2} \right| = \left| \frac{x-4}{3} \right| < 1$

$|x-4| < 3$ $R=3$

$\boxed{(1, 7]}$

$\boxed{\text{oops, I went overboard. } R = 3}$

$x=1$ $\sum_0^\infty \frac{(-1)^n(-3)^n}{3^n(n+1)} = \sum_0^\infty \frac{\cancel{(-1)^n (-1)^n} \cancel{3^n}}{\cancel{3^n} (n+1)} = \sum_0^\infty \frac{1}{n+1}$

Diverges → compare to harm

$x=7$ $\sum_0^\infty \frac{(-1)^n \cancel{3^n}}{\cancel{3^n}(n+1)} = \sum_0^\infty \frac{(-1)^n}{n+1}$ converge Alt. series $\frac{1}{n+1}$ dec an going to 0

(c) $1 - \frac{1}{3(2)}(5-4) + \frac{1}{3^2(3)}(5-4)^2 - \frac{1}{3^3(4)}(5-4)^3 + \ldots$

~~Error < $\frac{2}{100}$~~

error < $\frac{1}{50}$

$1 - \frac{1}{6} + \frac{1}{27} - \frac{1}{108}$

this is alternating series remainder term...

$1 - \frac{1}{6} + \frac{1}{27} = \boxed{.870}$

pdf answer key seems wrong again